Iterated forcing, Part 3: FS iteration and small forcings

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Outline



2 Background

- 3 Small forcings
- 4 The left side of Cichoń's diagram

Outline

1 Postscript to Lecture 2

2 Background

- **3** Small forcings
- 4 The left side of Cichoń's diagram

A correction concerning yesterday's talk. I claimed that a *V*-generic filter $G \subseteq \prod_i Q_i$ will induce filters $G(i) \subseteq Q_i \dots$

- ... which are *V*-generic subsets of *Q_i*. (That is true!)
- ... which are not *V*[*G*(*j*)]-generic. (That was false, as several of you have pointed out.)

- The forcing notions Q_i are given by definitions in V. ("Set of all sequences")
- The same definition will give a forcing notion Q_i in V[G(j)].
- $V[G(I)]\models Q_{I}
 eq Q_{I}'$ in general, and often not even $Q_{I}lpha Q_{I}'$.
- The filter G(I) will be Qr-generic over V[G(I)], but in general not Qr-generic.

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Definition

An iteration (P_{α} , Q_{α} : $\alpha < \delta$) is called a FS iteration iff:

- For each limit ε < δ of cofinality ω, Pε is the direct limit of (Pα, Qα : α < ε) (i.e., Pε = Uα<ε Pα).
- Equivalently: Each P_β is the set of all partial functions p with finite domain ⊆ β, s.t. for all α: p↾α ⊨ p(α) ∈ Q_α.

For any such (topless) iteration we define its finite support limit P_{δ} as the direct limit. We write \Vdash_{α} instead of $\Vdash_{P_{\alpha}}$.

Theorem

If for all α < δ we have ⊩_α Q_α ⊨ ccc, then also P_δ ⊨ ccc.
 If for all n < ω we have Q_α K ccc, then ⊩_ω ω ≈ ω²_δ.
 ("P_α collapses N₁.")

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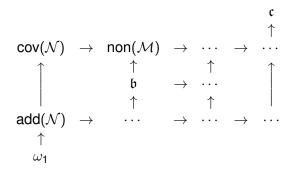
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A fragment of Cichoń's Diagram



- How many N sets (=sets of Lebesgue measure 0, null sets) do we have to add together (in the sense of ∪) to get a non-null set?
- How many null sets do we need to cover the real line?
- How many points do we need to get a non-Meager set?

• . . .

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Definition

We write \mathbb{B} for random forcing. \mathbb{B} adds a real that avoids every Borel measure zero set whose code is in the ground model. We write \mathbb{D} for Hechler forcing. \mathbb{D} adds a function in ω^{ω} which dominates all old functions.

Fact

Let λ be regular uncountable. Let $(P_{\alpha}, Q_{\alpha} : \alpha < \lambda)$ be an iteration where cofinally often we have $Q_{\alpha} = \mathbb{B} =$ random forcing. Then $\Vdash_{\lambda} \operatorname{cov}(\mathcal{N}) \geq \lambda$.

Proof.

Every small family of null sets appears in an intermediate model (use ccc!), so the next random real is not covered.

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Let κ be a cardinal, P a forcing notion. A κ -subforcing of P is a nice subset of P of size κ , typically $P \cap N$ for some elementary model, or $P \cap V_0$ for some "earlier" model V_0 in an iteration.

Subforcings must agree on \leq and $\perp,$ so subforcings of ccc forcings are again ccc.

- Let λ be regular uncountable, $\kappa_{cn} \leq \lambda$. Let $(P_{\alpha}, Q_{\alpha} : \alpha < \lambda)$ be an iteration where every $< \kappa_{cn}$ -sized subforcing of \mathbb{B} appears somewhere as Q_{α} . Then \mathbb{H}_{λ} cov $(\mathcal{N}) \geq \kappa_{cn}$.
- Similarly: Let κ₀ ≤ λ. Let (P₀, Q₀ : α < λ) be an iteration where every < κ₀-sized subforcing of 0 appears somewhere as Q₀......Then l₁ t₁ ≥ κ₀.
 - Combining these two constructions yields a model of σ $\sigma\sigmar(N) = rer \leq r_0 = 0$

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- Let λ be regular uncountable, κ_{cn} ≤ λ.
 Let (P_α, Q_α : α < λ) be an iteration where every
 < κ_{cn}-sized subforcing of B appears somewhere as Q_α.
 Then ⊩_λ cov(N) ≥ κ_{cn}.
- Similarly: Let κ_b ≤ λ. Let (P_α, Q_α : α < λ) be an iteration where every < κ_b-sized subforcing of D appears somewhere as Q_α. — Then H_λ b ≥ κ_b.
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Definition A sequence $\vec{f} = (f_i : i < \kappa)$ is a scale in $(\omega^{\omega}, \leq^*)$ if

• For all i < j we have $f_i \leq^* f_j$.

• \overline{f} is unbounded, i.e.: there is no g with $\forall i : f_i \leq^* g$. (Note: \overline{f} is not necessarily dominating.) Recall that b is the shortest length of a scale. Theorem (How to keep $b \leq \kappa$) Let $\overline{f} = (f_i : i < \kappa)$ be a scale in (ω^{μ}, \leq^*) , κ regular. Then. • If Q is a forcing of size $< \kappa$, then \Vdash_Q \overline{f} is a scale".

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Theorem (How to keep $b \leq \kappa$)

Let $\tilde{f} = (f_i : i < \kappa)$ be a scale in $(\omega^{\omega}, \leq^*)$, κ regular. Then:

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An ambitious plan

Does the following plan work? To get a model where a specific cardinal \mathfrak{x} has value κ , and the continuum has value λ , try this:

- Find a (nice) forcing notion Q which "increases x".
 (Nice = Souslin ccc, i.e.: the relations ≤_Q and also ⊥_Q are analytic often even Borel)
- (For example, if r = cov(I), where I is a Borel ideal, find a forcing such that the generic object is in no set from I which comes from the ground model. Similar to random/Hechler from before.)
- Use an iteration (P_α, Q_α : α < λ) where each < λ-sized subforcing of Q appears as some Q_α.
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Examples (of failure)

- There are no "small subforcings of Cohen". Every iteration of length λ (λ regular ≥ c) will force cov(M) = λ.
- Subforcings of nice forcings can be very naughty. For example, there may be a subforcing of 8 which adds a dominating real. (Even though 8 Is w^e-bounding.)

Example (of success)

Let $(P_{\alpha}, Q_{\alpha} : \alpha < \aleph_{\omega+1})$ be an iteration in which each Q_{α} is a (cleverly chosen) subforcing of \mathbb{B} of size $< \aleph_{\omega}$. Then $\Vdash_{\aleph_{\omega+1}} \operatorname{cov}(\mathcal{N}) = \aleph_{\omega}$. (!!!)

- Difficult.
- In contrast, $cov(\mathcal{M})$ must have cofinality $\geq add(\mathcal{N}) \geq \omega_1$.

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Let $(P_{\alpha}, Q_{\alpha} : \alpha < \aleph_{\omega+1})$ be an iteration in which each Q_{α} is a (cleverly chosen) subforcing of \mathbb{B} of size $< \aleph_{\omega}$. Then $\Vdash_{\aleph_{\omega+1}} \operatorname{cov}(\mathcal{N}) = \aleph_{\omega}$. (!!!)

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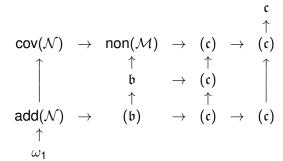
Outline

- Postscript to Lecture 2
- 2 Background
- **3** Small forcings
- 4 The left side of Cichoń's diagram

Many different cardinals

Theorem (G-Mejía-Shelah)

There is a model in which all the displayed cardinals have different values.



This model can be obtained using the technique of "small" forcings. However, the small forcing notions increasing non(\mathcal{M}) have to be chosen carefully, as they threaten to add dominating reals and hence increase \mathfrak{b} .

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$$\begin{array}{cccc} & & & & & \uparrow \\ \mathsf{cov}(\mathcal{N}) & \to & \mathsf{non}(\mathcal{M}) & \to & (\mathfrak{c}) & \to & (\mathfrak{c}) \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \mathfrak{b} & \to & (\mathfrak{c}) & & \uparrow \\ & & \uparrow & & \uparrow & & \uparrow \\ \mathsf{add}(\mathcal{N}) & \to & (\mathfrak{b}) & \to & (\mathfrak{c}) & \to & (\mathfrak{c}) \\ & \uparrow & & & & & & \\ \end{array}$$

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Definition

The forcing notion \mathbb{E} is the set of all conditions $p = (s^p, w^p, \varphi^p)$ where:

- $\boldsymbol{s} \in \omega^{<\omega}$
- $\mathbf{W} \in \boldsymbol{\omega}$
- φ = (φ_k : k ∈ ω) is a family of sets in [ω]^{≤w} (a "slalom" of bounded width w)
- $\forall i < |s| : s_i \notin \varphi_i$

The generic object g will be a sequence in ω^{ω} . p forces that g extends s and avoids all sets in φ : $g(i) \notin \varphi_i$. g defines a meager set $M_g = \{x \in \omega^{\omega} : \forall^{\infty} i x(i) \neq g(i)\}$ ("eventually different"). Every real from the ground model will be in M_g .

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Your patience will be rewarded.